Mathematical Methods – Aerospace Engineering (Forlì)
Complementi di Analisi Matematica M – Laurea Magistrale in Ingegneria per l'Ambiente e il Territorio e Ingegneria Chimica e di Processo – Prof. A. Bonfiglioli

Exercises on the Laplace Transform

► **Exercise** 1. Solve the following Cauchy problems, by using the well-known Laplace-transform method and Residue Calculus computations; verify –*a posteriori*– that what you obtained is indeed the solution of the Cauchy problem.

1. $\begin{cases} u'' - 3\,u' + 2\,u = 2\,\cos t + 6\,\sin t \\ u(0) = 2 \\ u'(0) = -1 \end{cases}$ [Solution: $e^t - e^{2t} + 2\cos t$]

2. $\begin{cases} u'' - 3\,u' + 2\,u = 2\,t^2 - 6\,t + 2 \\ u(0) = 3 \\ u'(0) = 3 \end{cases}$ [Solution: $t^2 + 3\,e^t$]

3. $\begin{cases} u'' - 6\,u' + 9\,u = 9\,t - 6 \\ u(0) = 1 \\ u'(0) = 4 \end{cases}$ [Solution: $t + e^{3t}$]

4. $\begin{cases} u'' - 6\,u' + 9\,u = e^{-t} \\ u(0) = 1 \\ u'(0) = -1 \end{cases}$ [Solution: e^{-t}]

5. $\begin{cases} u'' + 2\,u' + u = 6\,t\,e^{-t} \\ u(0) = 0 \\ u'(0) = 0 \end{cases}$ [Solution: $t^3\,e^{-t}$]

6. $\begin{cases} u'' - 4\,u' + 4\,u = e^t \\ u(0) = 3 \\ u'(0) = 5 \end{cases}$ [Solution: $e^t + 2\,e^{2t}$]

7. $\begin{cases} u'' - 2\,u' + 10\,u = 40 \\ u(0) = 4 \\ u'(0) = 3 \end{cases}$ [Solution: $4 + e^t\,\sin(3t)$]

8. $\begin{cases} u'' - 2\,u' + 10\,u = 0 \\ u(0) = -2 \\ u'(0) = 1 \end{cases}$ [Solution: $e^t \sin(3t) - 2\,e^t\cos(3t)$]

9. $\begin{cases} u'' - 4\,u' + 4\,u = 4(t+3) \\ u(0) = 7 \\ u'(0) = 6 \end{cases}$ [Solution: $3\,e^{2t} - t\,e^{2t} + 4 + t$]

10. $\begin{cases} u'' - 4\,u = e^t(1+6t) \\ u(0) = 1 \\ u'(0) = 7 \end{cases}$ [Solution: $e^{2t} - e^{-2t} + e^t + 2\,t\,e^t$]

11. $\begin{cases} u'' + 2\,u' - 3\,u = 6\,t\,e^{-t} \\ u(0) = 0 \\ u'(0) = 1 \end{cases}$ [Solution: $-\frac{5}{8}\,e^{-3t} + \frac{5}{8}\,e^t - \frac{3}{2}\,t\,e^{-t}$]

12. $\begin{cases} u'' + 4\,u' + 4\,u = 9\,e^t \\ u(0) = 4 \\ u'(0) = -6 \end{cases}$ [Solution: $3\,e^{-2t} - t\,e^{-2t} + e^t$]

13. $\begin{cases} u'' - u' - 2\,u = -6\,e^{-t} \\ u(0) = -4 \\ u'(0) = 3 \end{cases}$ [Solution: $-e^{2t} - 3\,e^{-t} + 2\,t\,e^{-t}$]

14. $\begin{cases} u'' - 6\,u' + 9\,u = e^{-t} \\ u(0) = 1 \\ u'(0) = -1 \end{cases}$ [Solution: $\frac{1}{16}\,e^{-t} + \frac{5}{16}\,e^{3t} - \frac{15}{4}\,t\,e^{3t}$]

15. $\begin{cases} u'' - 4\,u' + 4\,u = 4\,t + 12 \\ u(0) = 7 \\ u'(0) = 6 \end{cases}$ [Solution: $4 + t + 3\,e^{2t} - t\,e^{2t}$]

16. $\begin{cases} u'' - 4\,u' + 4\,u = e^t \\ u(0) = 3 \\ u'(0) = 5 \end{cases}$ [Solution: $e^t + 2\,e^{2t}$]

You may want to use some of the following formulas:

(I). Elementary functions:

1. $L\big[1\big](s) = \dfrac{1}{s}$
2. $L\big[t^n\big](s) = \dfrac{n!}{s^{n+1}}$ (with $n \in \mathbb{N}$)
3. $L\big[e^{z\,t}\big](s) = \dfrac{1}{s-z}$ (with $z \in \mathbb{C}$)
4. $L\left[\dfrac{t^{n-1}e^{z\,t}}{(n-1)!}\right](s) = \dfrac{1}{(s-z)^n}$ (with $z \in \mathbb{C}$ and $n \in \mathbb{N}$)
5. $L\big[\sin(\omega\,t)\big](s) = \dfrac{\omega}{s^2+\omega^2}$ (with $\omega \in \mathbb{R}$)
6. $L\big[\cos(\omega\,t)\big](s) = \dfrac{s}{s^2+\omega^2}$ (with $\omega \in \mathbb{R}$)
7. $L\big[\sinh(\omega\,t)\big](s) = \dfrac{\omega}{s^2-\omega^2}$ (with $\omega \in \mathbb{R}$)
8. $L\big[\cosh(\omega\,t)\big](s) = \dfrac{s}{s^2-\omega^2}$ (with $\omega \in \mathbb{R}$)

(II). Rescaling/translating:

1. $L\big[f(c\,t)\big](s) = \dfrac{1}{c}\,L\big[f\big]\left(\dfrac{s}{c}\right)$ (with $c > 0$)
2. $L\Big[f_+(t-t_0)\Big](s) = e^{-t_0 s}\,L\big[f\big](s)$ (with $t_0 > 0$)
3. $L\Big[e^{zt}\,f(t)\Big](s) = L\big[f\big](s-z)$ (with $z \in \mathbb{C}$)

(III). Derivatives:

1. $L\big[u'\big](s) = s\,L\big[u\big](s) - u(0)$
2. $L\Big[u^{(n)}\Big](s) = s^n\,L\big[u\big](s) - \Big(s^{n-1}u(0) + s^{n-2}u'(0) + \cdots + s\,u^{(n-2)}(0) + u^{(n-1)}(0)\Big)$ (with $n \in \mathbb{N}$)
3. $\left(\dfrac{\mathrm{d}}{\mathrm{d}s}\right) L\big[u\big](s) = -L\Big[t\,u(t)\Big](s)$
4. $\left(\dfrac{\mathrm{d}}{\mathrm{d}s}\right)^n L\big[u\big](s) = (-1)^n\,L\Big[t^n\,u(t)\Big](s)$ (with $n \in \mathbb{N}$)

(IV). Convolution:

1. $L\big[f * g\big](s) = L\big[f\big](s) \cdot L\big[g\big](s)$
2. $L\left[\displaystyle\int_0^t g\right](s) = \dfrac{1}{s} \cdot L\big[g\big](s)$